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On the Log-Concavity of the Root of the Catalan-Larcombe-French Numbers

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Abstract: Recently, some combinatorial properties for the the Catalan-Larcombe-French numbers have been proved by Sun and Wu, and Zhao. Recently, Z. W. Sun conjectured that the root of the Catalan-Larcombe-French numbers is log-concave. In this paper, we confirm Sun's conjecture by establishing the lower and upper bound for the ratios of the Catalan-Larcombe-French numbers.

Keywords: The Catalan-Larcombe-French Number, Log-Concavity, Recurrence Relation

1. Introduction

The aim of this paper is to prove the log-concavity of $\{\sqrt[n]{P_n}\}_{n=1}^{\infty}$, where P_n is the n-th Catalan-Larcombe-French number. This confirms a conjecture given by Zhi Wei Sun [8].

Recall that an infinite sequence a_n is said to be log-concave if for $n\geq 1$,

$$a_n^2 \geq a_{n-1}a_{n+1}$$

The Catalan-Larcombe-French numbers P_n were first defined by Catalan in terms of the “Segner numbers”. The n-th Catalan-Larcombe-French number is generated by either of the finite sums

$$P_n = 2^n \sum_{p=0}^{[n/2]} (-4)^p \binom{2(n-p)}{n-p}^2 \binom{n-p}{p}, n \geq 0$$

(where the function $[x]$, for arbitrary x real, is the greatest integer not exceeding x) or

$$P_n = \frac{1}{n!} \sum_{p+q=n} \binom{2p}{p}\binom{2q}{q} \frac{(2p)!\,(2q)!}{p!\,q!}, n \geq 0$$

The infinite sequence $\{P_n\}_{n=0}^{\infty}$ is known as the Catalan-Larcombe-French sequence (Sequence No. A053175 in Sloane's database [7]). These numbers occur in the theory of elliptic integrals, and there are relations to the arithmetic-geometric-mean. Furthermore, the sequence satisfies the following recurrence relation:

$$P_n = \frac{8(3n^2-3n+1)}{n^2} P_{n-1} - \frac{128(n-1)^2}{n^2} P_{n-2} \tag{1}$$

for $n\geq 2$, with the initial values given by $P_0 = 1$ and $P_1 = 8$. For more details, see [1-6].

The combinatorial properties for the Catalan-Larcombe-French sequence have been considered. The log-behavior of the Catalan-Larcombe-French sequence was studied by Zhao [12]. Moreover, she proved that the sequence $\{P_n\}_{n=0}^{\infty}$ is log-balanced. Xia and Yao [11] proved that the sequences $\left\{\frac{P_{n+1}}{P_n}\right\}_{n=0}^{\infty}$ and $\{\sqrt[n]{P_n}\}_{n=0}^{\infty}$ are strictly increasing. The 2-log-convexity of the sequence $\{P_n\}_{n=0}^{\infty}$ was proved by Sun and Wu [8]. Furthermore, Sun and Jin [9] proved the log-concavity for the sequence $\left\{\frac{P_n}{P_{n-1}}\right\}_{n=1}^{\infty}$, which confirmed a conjecture due to Sun and Wu [8].

Recently, Sun [10] posed the following conjecture:

Conjecture 1.1. *The sequence* $\{\sqrt[n]{P_n}\}_{n=1}^{\infty}$ *is log-concave. Moreover, the sequence* $\{\sqrt[n+1]{P_{n+1}}/\sqrt[n]{P_n}\}_{n=1}^{\infty}$ *is strictly decreasing.*

In this paper, we will establish lower and upper bounds for $\frac{P_n}{P_{n-1}}$ by utilizing the recurrence relation of P_n and then present a proof of Conjecture 1.1.

2. Lower and Upper Bounds for $\frac{P_n}{P_{n-1}}$

In order to prove Conjecture 1.1, we need to prove several inequalities for The Catalan-Larcombe-French numbers P_n and establish lower and upper bounds for $\frac{P_n}{P_{n-1}}$. We first establish a lower bound for $\frac{P_n}{P_{n-1}}$.

Lemma 2.1. *For n ≥7, we have*

$$f(n) < \frac{P_n}{P_{n-1}} \tag{2}$$

where

$$f(n) = \frac{16(8n^3-20n^2+14n-11)}{(2n-1)^3} \tag{3}$$

Proof. We are ready to prove Lemma 2.1 by induction on n. It is easy to check that Inequality (2) is true when $n = 7$. Suppose that Lemma 2.1 holds when $n = m \geq 7$, namely,

$$\frac{16(8m^3-20m^2+14m-11)}{(2m-1)^3} < \frac{P_m}{P_{m-1}} \tag{4}$$

In order to prove this lemma, we need to prove that (2) holds when $n = m+1$, that is,

$$\frac{16(8(m+1)^3-20(m+1)^2+14(m+1)-11)}{(2(m+1)-1)^3} < \frac{P_{m+1}}{P_m} \tag{5}$$

Thanks to (1) and (4), we deduce that

$$\frac{P_{m+1}}{P_m} = \frac{8(3(m+1)^2-3(m+1)+1)}{(m+1)^2} - \frac{128m^2}{(m+1)^2}\frac{P_{m-1}}{P_m}$$
$$> \frac{8(3(m+1)^2-3(m+1)+1)}{(m+1)^2} - \frac{128m^2}{(m+1)^2}\frac{(2m-1)^3}{16(8m^3-20m^2+14m-11)}$$
$$= \frac{8(16m^5-24m^4-16m^3-10m^2-19m-11)}{(m+1)^2(8m^3-20m^2+14m-11)} \tag{6}$$

Thanks to (6), we have

$$\frac{P_{m+1}}{P_m} - \frac{16(8(m+1)^3-20(m+1)^2+14(m+1)-11)}{(2(m+1)-1)^3}$$
$$> \frac{8(16m^5-24m^4-16m^3-10m^2-19m-11)}{(m+1)^2(8m^3-20m^2+14m-11)} - \frac{16(8(m+1)^3-20(m+1)^2+14(m+1)-11)}{(2(m+1)-1)^3}$$
$$= \frac{8(32m^6-80m^5-384m^4-488m^3-254m^2-273m-209)}{(m+1)^2(8m^3-20m^2+14m-11)(2m+1)^3} > 0. \tag{7}$$

Inequality (5) follows from (7). Therefore, Lemma 2.1 is proved by induction. The proof is complete.

We are now in a position to establish an upper bound for $\frac{P_n}{P_{n-1}}$.

Lemma 2.2. For n ≥0, we have

$$\frac{P_n}{P_{n-1}} < f(n+1), \tag{8}$$

where $f(n)$ is defined by (3).

Proof. We also prove Lemma 2.2 by induction on n. It is easy to verify that (8) is true when $n = 7$. Assume that Lemma 2.2 holds when $n = m \geq 7$, namely,

$$\frac{P_m}{P_{m-1}} < \frac{16(8(m+1)^3-20(m+1)^2+14(m+1)-11)}{(2(m+1)-1)^3}. \tag{9}$$

In order to prove (8), it suffices to prove that (8) is true when $n = m + 1$, that is,

$$\frac{P_{m+1}}{P_m} < \frac{16(8(m+2)^3-20(m+2)^2+14(m+2)-11)}{(2(m+2)-1)^3} \tag{10}$$

It follows from (1) and (9) that

$$\frac{P_{m+1}}{P_m} = \frac{8(3(m+1)^2-3(m+1)+1)}{(m+1)^2} - \frac{128m^2}{(m+1)^2}\frac{P_{m-1}}{P_m}$$
$$< \frac{8(3(m+1)^2-3(m+1)+1)}{(m+1)^2} - \frac{128m^2}{(m+1)^2}\frac{(2(m+1)-1)^3}{16(8(m+1)^3-20(m+1)^2+14(m+1)-11)}$$
$$= \frac{8(16m^5+24m^4+8m^3-30m^2-29m-9)}{(m+1)^2(8m^3+4m^2-2m-9)}. \tag{11}$$

Therefore, by (11), we deduce that

$$\frac{P_{m+1}}{P_m} - \frac{16(8(m+2)^3-20(m+2)^2+14(m+2)-11)}{(2(m+2)-1)^3}$$
$$< \frac{8(16m^5+24m^4+8m^3-30m^2-29m-9)}{(m+1)^2(8m^3+4m^2-2m-9)} - \frac{16(8(m+2)^3-20(m+2)^2+14(m+2)-11)}{(2(m+2)-1)^3}$$
$$= -\frac{8(32m^6+80m^5+288m^4+744m^3+978m^2+689m+225)}{(m+1)^2(8m^3+4m^2-2m-9)(2m+3)^3} < 0. \tag{12}$$

Inequality (5) follows from (12) and Lemma 2.2 is proved by induction. This completes the proof.

3. Proof of Conjecture 1.1

In this section, we provide a proof of Conjecture 1.1 by utilizing the lower and upper bounds for $\frac{P_n}{P_{n-1}}$ established in Section 2. We first prove the following lemma:

Lemma 3.1. *For n ≥7,*

$$\left(1-\frac{2}{n^2+n+2}\right)^{n^2+n+2} \geq \left(1-\frac{1}{29}\right)^{58} \tag{13}$$

Proof. It is easy to see that for $x > y > 0$,

$$x^{n+1}-y^{n+1} = (x-y)(x^n+x^{n-1}y+\cdots+xy^{n-1}+y^n) > (n+1)(x-y)y^n. \tag{14}$$

If we set $x = 1-\frac{2}{n+1}$ and $y = 1-\frac{2}{n}$ in (14), then we get

$$\left(1-\frac{2}{n+1}\right)^{n+1}-\left(1-\frac{2}{n}\right)^{n+1}>$$

$$(n+1)\left(\frac{2}{n}-\frac{2}{n+1}\right)\left(1-\frac{2}{n}\right)^{n}=\frac{2}{n}\left(1-\frac{2}{n}\right)^{n}. \quad (15)$$

Therefore, it follows from (15) that

$$\left(1-\frac{2}{n+1}\right)^{n+1}>\left(1-\frac{2}{n}\right)^{n+1}+\frac{2}{n}\left(1-\frac{2}{n}\right)^{n}$$

$$=\left(1-\frac{2}{n}+\frac{2}{n}\right)\left(1-\frac{2}{n}\right)^{n}=\left(1-\frac{2}{n}\right)^{n} \quad (16)$$

Therefore, the sequence $\left\{\left(1-\frac{2}{n}\right)^{n}\right\}_{n=2}^{\infty}$ is increasing. In particular, its subsequence $\left\{\left(1-\frac{2}{n^2+n+2}\right)^{n^2+n+2}\right\}_{n=7}^{\infty}$ is also increasing. Hence, for $n \geq 7$,

$$\left(1-\frac{2}{n^2+n+2}\right)^{n^2+n+2}\geq\left(1-\frac{2}{7^2+7+2}\right)^{7^2+7+2}$$

$$=\left(1-\frac{1}{29}\right)^{58}. \quad (17)$$

which is nothing but (17). The proof of this lemma is complete.

We are now in a position to turn to prove Conjecture 1.1.

Proof of Conjecture 1.1. It is easy to check that Conjecture 1.1 is true when $1\leq n\leq 6$. Hence, we only need to consider the case $n \geq 7$. By (2) and (8), we deduce that for $n \geq 7$,

$$\left(\frac{P_{n+1}}{P_n}\right)^{(n+1)(n+2)}>f^{(n+1)(n+2)}(n+1) \quad (18)$$

$$\frac{P_{n+1}}{P_n}<f(n+2)<\frac{P_{n+2}}{P_{n+1}}<f(n+3) \quad (19)$$

and

$$\frac{P_8}{P_7}<f(9)<\frac{P_9}{P_8}<f(10)<\cdots$$

$$<\frac{P_n}{P_{n-1}}<f(n+1)<\frac{P_{n+1}}{P_n}, \quad (20)$$

where f(n) is defined by (3). By (20), we see that for $n \geq 7$,

$$P_{n+1}^2=P_7^2\left(\frac{P_8}{P_7}\frac{P_9}{P_8}\cdots\frac{P_n}{P_{n-1}}\right)^2\frac{P_{n+1}^2}{P_n^2}\leq P_7^2f^{2n-14}(n+1)\frac{P_{n+1}^2}{P_n^2}. \quad (21)$$

In view of (19) and (21), we see that

$$\left(\frac{P_{n+2}}{P_{n+1}}\right)^{n(n+1)}P_{n+1}^2<f^{n(n+1)}(n+3)P_{n+1}^2$$

$$<f^{n^2+n}(n+3)P_7^2f^{2n-14}(n+1)\frac{P_{n+1}^2}{P_n^2}$$

$$<f^{n^2+n+2}(n+3)P_7^2f^{2n-14}(n+1). \quad (22)$$

It is easy to check that for $n \geq 7$,

$$\frac{f(n+1)}{f(n+3)}-\left(1-\frac{2}{n^2+n+2}\right)$$

$$=\frac{2(64n^5+112n^4-576n^3-1992n^2-2196n-1125)}{(2n+1)^3(8n^3+52n^2+110n+67)(n^2+n+2)}>0. \quad (23)$$

Based on (3.6), (3.10) and (3.11), we deduce that for $n \geq 7$,

$$\left(\frac{P_{n+1}}{P_n}\right)^{(n+1)(n+2)}-\left(\frac{P_{n+2}}{P_{n+1}}\right)^{n(n+1)}P_{n+1}^2>f^{(n+1)(n+2)}(n+1)-f^{n^2+n+2}(n+3)P_7^2f^{2n-14}(n+1)$$

$$=f^{2n-14}(n+1)f^{n^2+n+2}(n+3)\left(\left(\frac{f(n+1)}{f(n+3)}\right)^{n^2+n+2}f^{14}(n+1)-P_7^2\right)$$

$$>f^{2n-14}(n+1)f^{n^2+n+2}(n+3)\left(\left(1-\frac{2}{n^2+n+2}\right)^{n^2+n+2}f^{14}(n+1)P_7^2\right) \quad (24)$$

By (13), (24) and the fact that $f(7) < f(n + 1)$ for $n \geq 7$,

$$\left(\frac{P_{n+1}}{P_n}\right)^{(n+1)(n+2)}-\left(\frac{P_{n+2}}{P_{n+1}}\right)^{n(n+1)}P_{n+1}^2>f^{2n-14}(n+1)f^{n^2+n+2}(n+3)\left(\left(1-\frac{1}{29}\right)^{58}f^{14}(n+1)-P_7^2\right)$$

$$>f^{2n-14}(n+1)f^{n^2+n+2}(n+3)\left(\left(1-\frac{1}{29}\right)^{58}f^{14}(7)-P_7^2\right) \quad (25)$$

With Maple, it is easy to verify that

$$\left(1-\frac{1}{29}\right)^{58}f^{14}(7)-P_7^2>0 \quad (26)$$

Combining (3.13) and (3.14) yields

$$\left(\frac{P_{n+1}}{P_n}\right)^{(n+1)(n+2)}>\left(\frac{P_{n+2}}{P_{n+1}}\right)^{n(n+1)}P_{n+1}^2 \quad (27)$$

Inequality (27) can be rewritten as

$$P_{n+1}^{2n^2+4n}>P_n^{(n+1)(n+2)}P_{n+2}^{n(n+1)}$$

Therefore,

$$\left(P_{n+1}^{2n^2+4n}\right)^{\frac{1}{n(n+1)(n+2)}} > \left(P_n^{(n+1)(n+2)} P_{n+2}^{n(n+1)}\right)^{\frac{1}{n(n+1)(n+2)}}$$

which yields

$$P_{n+1}^{\frac{2}{n+1}} > P_n^{\frac{1}{n}} P_{n+2}^{\frac{1}{n+2}}$$

The above inequality can be rewritten as

$$\frac{\sqrt[n+1]{P_{n+1}}}{\sqrt[n]{P_n}} > \frac{\sqrt[n+2]{P_{n+2}}}{\sqrt[n+1]{P_{n+1}}} \quad (28)$$

It follows from (28) that the sequence $\left\{\sqrt[n]{P_n}\right\}_{n=1}^{\infty}$ is log-concave and the sequence $\left\{\sqrt[n+1]{P_{n+1}}/\sqrt[n]{P_n}\right\}_{n=1}^{\infty}$ is strictly decreasing. This completes the proof of Conjecture 1.1.

4. Conclusion

The Catalan-Larcombe-French numbers play important roles in combinatorics and number theory. Many combinatorial properties and congruence properties for the Catalan-Larcombe-French numbers have been proved. In this paper, by establishing several inequalities for the Catalan-Larcombe-French numbers, we obtain the lower and upper bound for the quotient $\frac{P_n}{P_{n-1}}$ and prove that the *n*th Root of the Catalan-Larcombe-French Numbers are log-concave, which confirm a conjecture given by Sun.

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